

Vibration Theory and Applications

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4.5 Shock Spectrum

A shock is a disturbance or excitation pulse of displacement, velocity, acceleration, or force, whose duration is short compared to the characteristic period of the system. The application of this shock pulse to a single-degree-of-freedom system (which for convenience we will refer to as an oscillator), results in a time response of the oscillator. The maximum value of the time response, for a given shock pulse, depends on the natural frequency and damping of the oscillator. The plot of the maximum response of the oscillator against the natural frequency of the oscillator is the shock spectrum of the disturbance.

As an example, for a base excitation of $\ddot{y}(t)$ the relative displacement z is given by Eq. (4.4-2) as

$$z = -\frac{1}{\omega} \int_0^t \ddot{y}(\xi) \sin \omega(t - \xi) d\xi$$

Thus its maximum value

$$z_{\max} = \left[-\frac{1}{\omega} \int_0^t \ddot{y}(\xi) \sin \omega(t - \xi) d\xi \right]_{\max} \quad (4.5-1)$$

plotted against ωt_1 , where t_1 is some characteristic pulse time, would represent a shock spectrum.

EXAMPLE 4.5-1. Determine the shock spectrum of a step function as a function of the damping.

Solution. The response of a damped oscillator to a step function of magnitude F_0 was found in Sec. 4.3 and given as Eq. (4.3-6), which is rewritten as

$$\frac{xk}{F_0} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\sqrt{1-\zeta^2}\omega_n t - \psi)$$

$$\tan \psi = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

By differentiating and setting the velocity equal to zero, we find the time t_p corresponding to the peak response.

$$\begin{aligned} \frac{\dot{x}k}{F_0} &= \omega_n e^{-\zeta\omega_n t} \left[\sin(\sqrt{1-\zeta^2}\omega_n t - \psi) + \frac{\zeta}{\sqrt{1-\zeta^2}} \cos(\sqrt{1-\zeta^2}\omega_n t - \psi) \right] \\ &= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\cos \psi \sin(\sqrt{1-\zeta^2}\omega_n t - \psi) + \sin \psi \cos(\sqrt{1-\zeta^2}\omega_n t - \psi) \right] \\ &= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin \sqrt{1-\zeta^2}\omega_n t \end{aligned}$$

Thus the time corresponding to the peak response is found from

$$\sin \sqrt{1-\zeta^2}\omega_n t_p = 0,$$

or

$$\omega_n t_p = \frac{\pi}{\sqrt{1-\zeta^2}}$$

Substituting this value into the displacement equation, the peak response becomes

$$\begin{aligned} \left(\frac{xk}{F_0}\right)_{\max} &= 1 - \frac{e^{-\pi\zeta/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \cos\left(\pi - \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}}\right) \\ &= 1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}} \end{aligned}$$

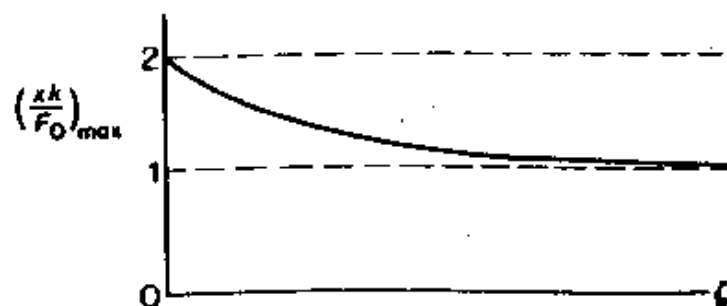


Fig. 4.5-1. Shock spectrum of a step function as a function of ζ .

We find in this case that the peak response is independent of the natural frequency ω_n of the oscillator and depends only on the damping ζ . A plot of this equation appears in Fig. 4.5-1.

EXAMPLE 4.5-2. Determine the undamped shock spectrum for a step function with a rise time t_1 , shown in Fig. 4.5-2.

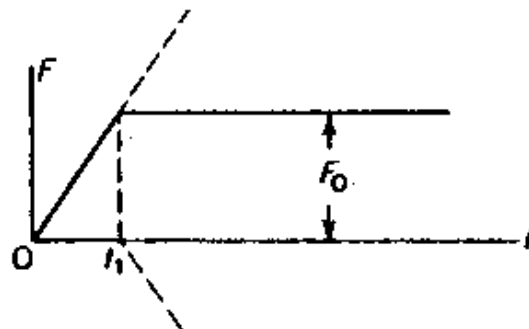


Fig. 4.5-2.

Solution. The input can be considered to be the sum of two ramp functions $F_0(t/t_1)$, the second of which is negative and delayed by the time t_1 . For the first ramp function the terms of Duhamel's equation are:

$$f(t) = F_0(t/t_1)$$

$$g(t) = \frac{1}{m\omega_n} \sin \omega_n t = \frac{\omega_n}{k} \sin \omega_n t$$

and the response becomes

$$\begin{aligned} x(t) &= \frac{\omega_n}{k} \int_0^t \frac{F_0 \xi}{t_1} \sin \omega_n(t - \xi) d\xi \\ &= \frac{F_0}{k} \left(\frac{t}{t_1} - \frac{\sin \omega_n t}{\omega_n t_1} \right), \quad t < t_1 \end{aligned}$$

For the second ramp function starting at t_1 , the solution can be written down by inspection of the above equation as

$$x(t) = -\frac{F_0}{k} \left[\frac{(t - t_1)}{t_1} - \frac{\sin \omega_n(t - t_1)}{\omega_n t_1} \right]$$

By superimposing these two equations the response for $t > t_1$ becomes

$$x(t) = \frac{F_0}{k} \left[1 - \frac{\sin \omega_n t}{\omega_n t_1} + \frac{1}{\omega_n t_1} \sin \omega_n(t - t_1) \right] \quad t > t_1$$

Differentiating and equating to zero, the peak time is obtained as

$$\tan \omega_n t_p = \frac{1 - \cos \omega_n t_1}{\sin \omega_n t_1}$$

Since $\omega_n t_p$ must be greater than π , we also obtain

$$\begin{aligned} \sin \omega_n t_p &= -\sqrt{\frac{1}{2}(1 - \cos \omega_n t_1)} \\ \cos \omega_n t_p &= \frac{-\sin \omega_n t_1}{\sqrt{2(1 - \cos \omega_n t_1)}} \end{aligned}$$

Substituting these quantities into $x(t)$, the peak amplitude is found as

$$\left(\frac{xk}{F_0}\right)_{\max} = 1 + \frac{1}{\omega_n t_1} \sqrt{2(1 - \cos \omega_n t_1)}$$

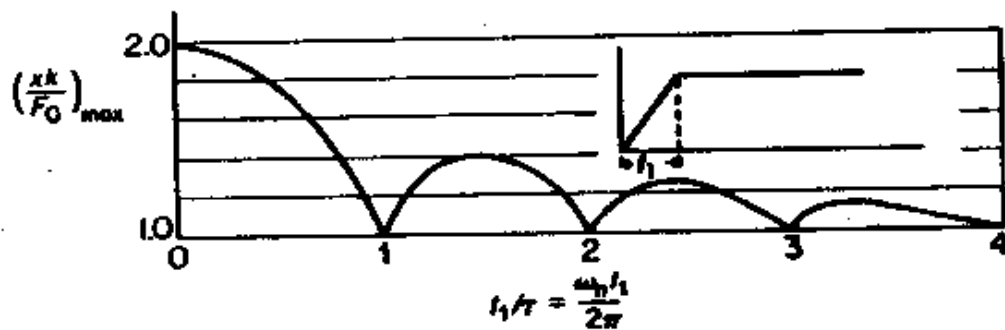


Fig. 4.5-3. Shock spectrum for step-ramp function.

Letting $\tau = 2\pi/\omega_n$ be the period of the oscillator, the above equation is plotted against t_1/τ in Fig. 4.5-3.

EXAMPLE 4.5-3. Determine the shock spectrum for the base velocity input, $\dot{y}(t) = v_0 e^{-t/t_0}$ of Example 4.4-1.

Solution. The relative displacement $z(t)$ was found in Example 4.4-1 to be

$$z(t) = \frac{v_0 t_0}{1 + (\omega_n t_0)^2} \times (e^{-t/t_0} - \omega_n t_0 \sin \omega_n t - \cos \omega_n t)$$

To determine the peak value z_p , the usual procedure is to differentiate the equation with respect to time t , set it equal to zero, and substitute this time back into the equation for $z(t)$. It is evident that for this problem this

results in a transcendental equation which must be solved by plotting. To avoid this numerical task, we will consider a different approach as follows.

For a very stiff system, which corresponds to large ω_n , the peak response will certainly occur at small t , and we would obtain for the time varying part of the equation the peak value

$$(1 - \omega_n t_0 - 1) \cong -\omega_n t_0$$

Thus for large ω_n the peak value will be nearly equal to

$$|z_p| \cong \frac{v_0 t_0}{1 + (\omega_n t_0)^2} (\omega_n t_0) \cong \frac{v_0 t_0}{\omega_n t_0}$$

so that $\frac{z_p}{v_0 t_0}$ plots against $\omega_n t_0$ as a rectangular hyperbola.

For small ω_n , or a very soft spring, the duration of the input would be small compared to the period of the system. Hence the input would appear as an impulsive doublet shown in Fig. 4.5-4 with the equation $v_0 t_0 \delta'(t)$. The solution for $z(t)$ is then

$$z(t) = v_0 t_0 \cos \omega_n t$$

and its peak value is

$$|z_p| \cong v_0 t_0$$

With these extreme conditions evaluated, we can now fill in the shock spectrum which is shown in Fig. 4.5-5.

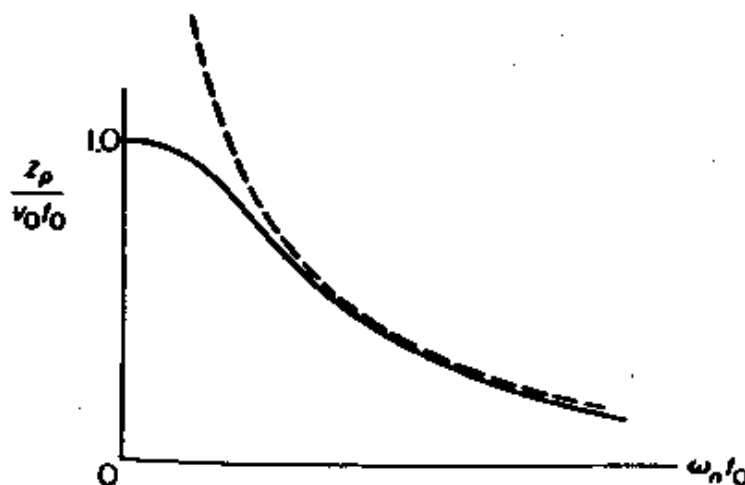


Fig. 4.5-5. Shock spectrum for the base velocity input $\dot{y}(t) = v_0 e^{-t/t_0}$.

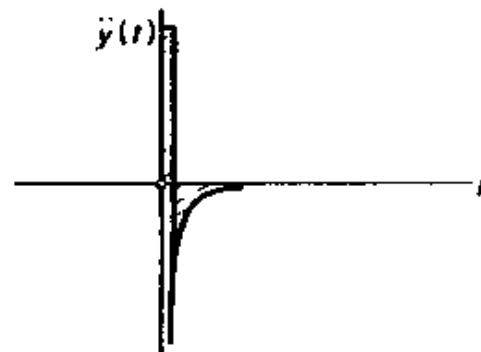
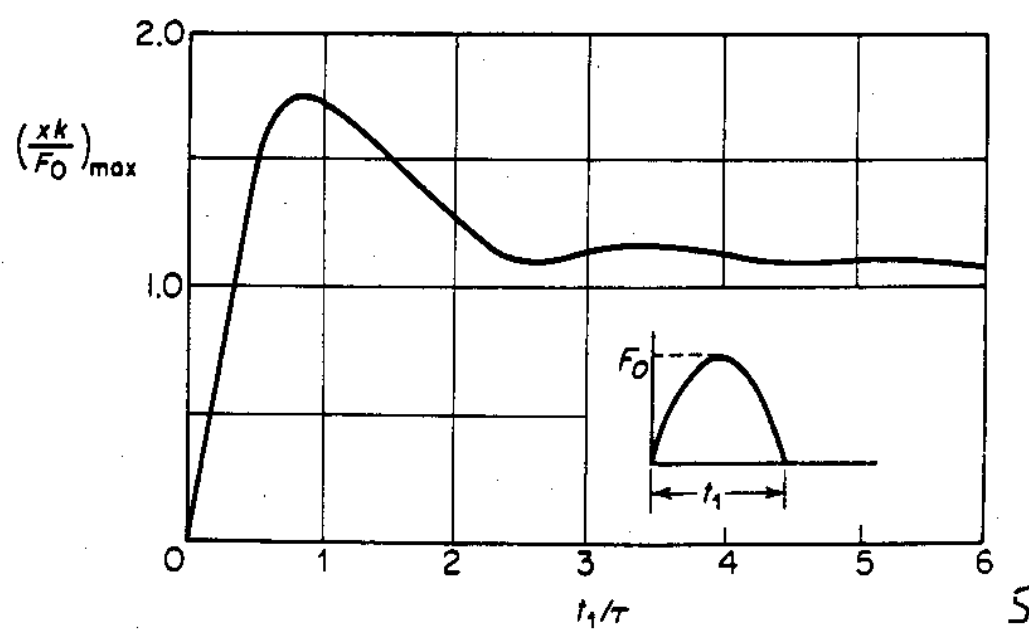
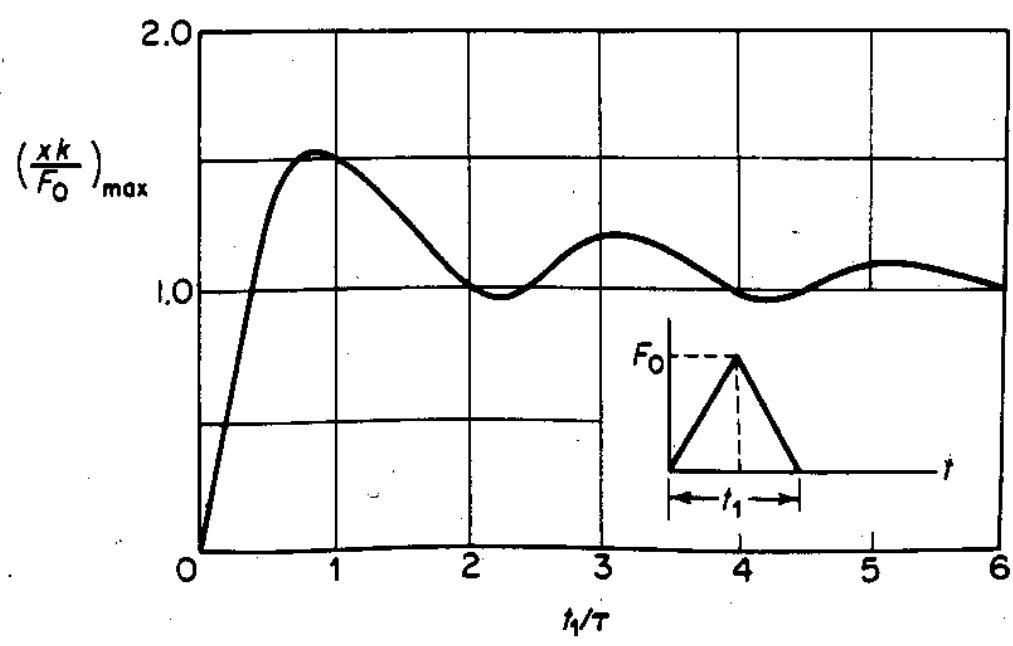
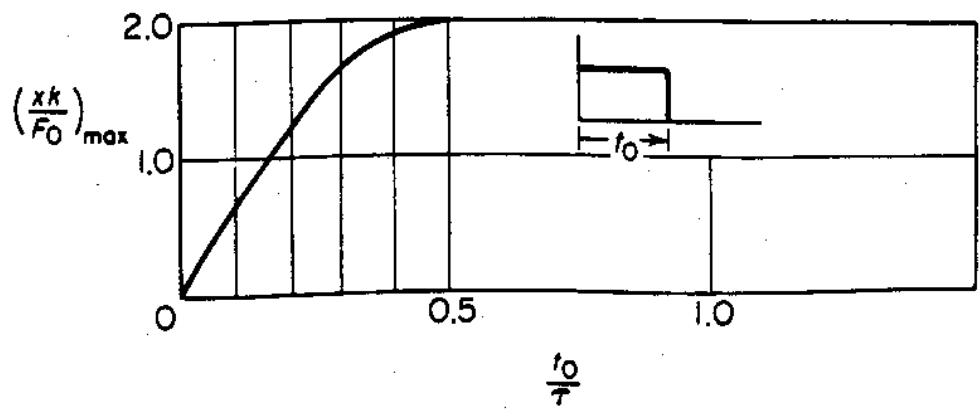
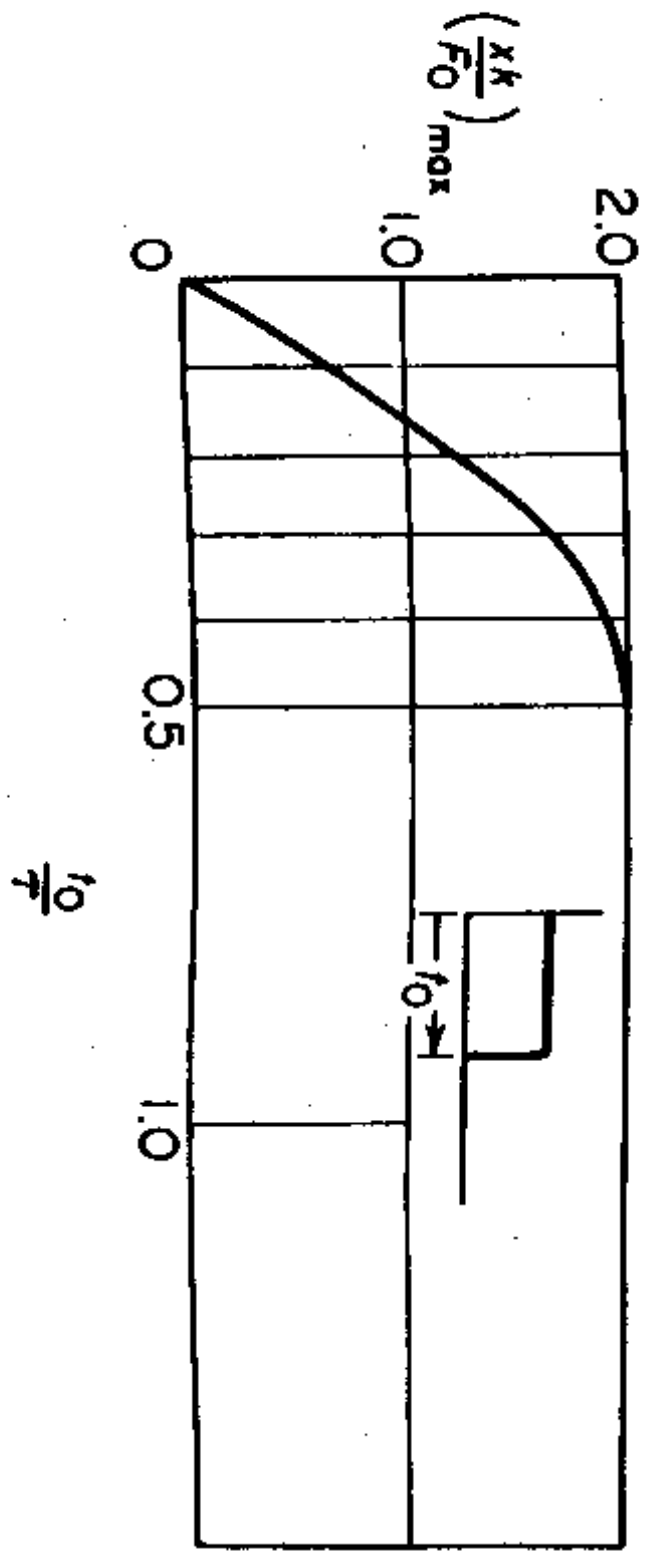
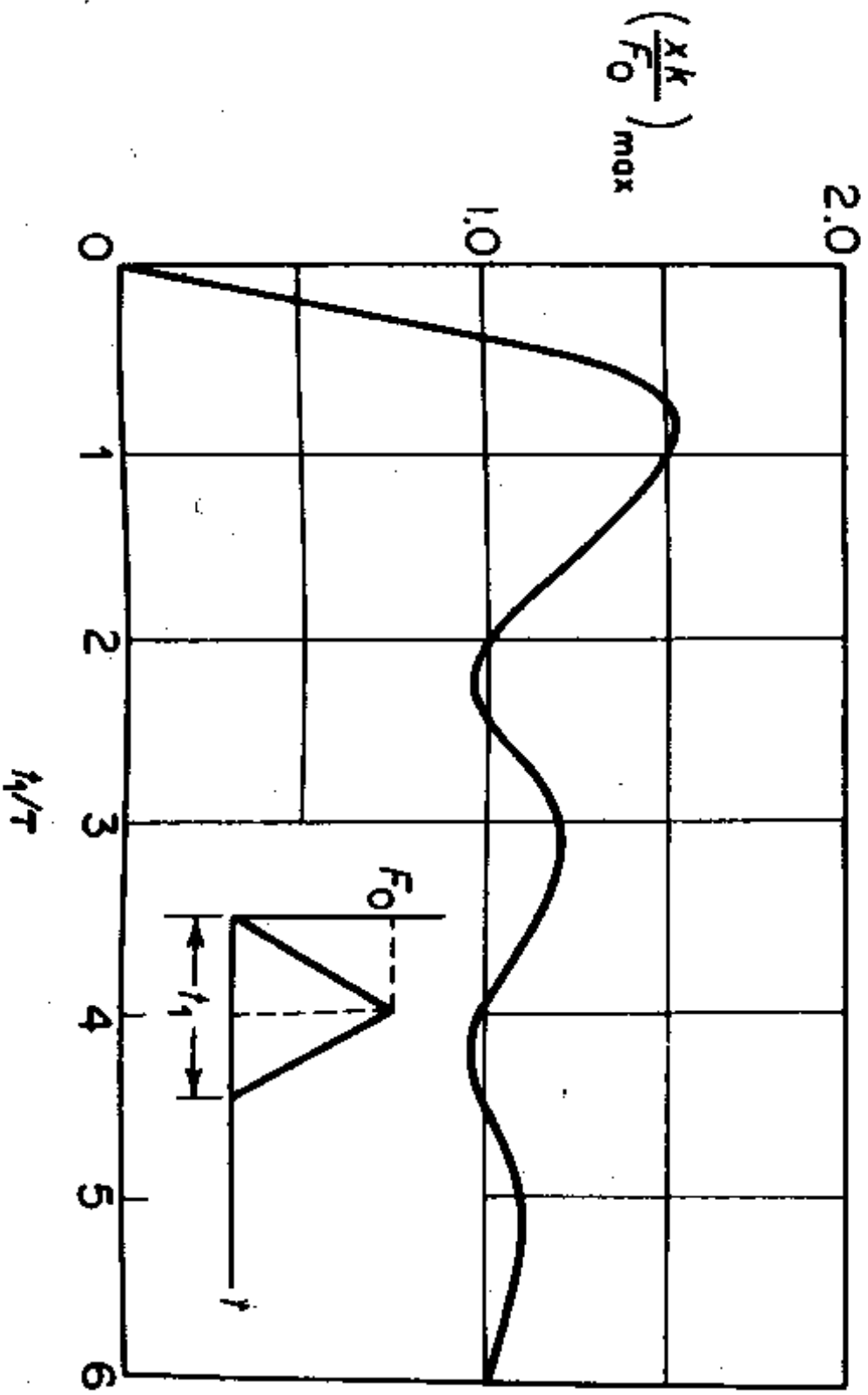


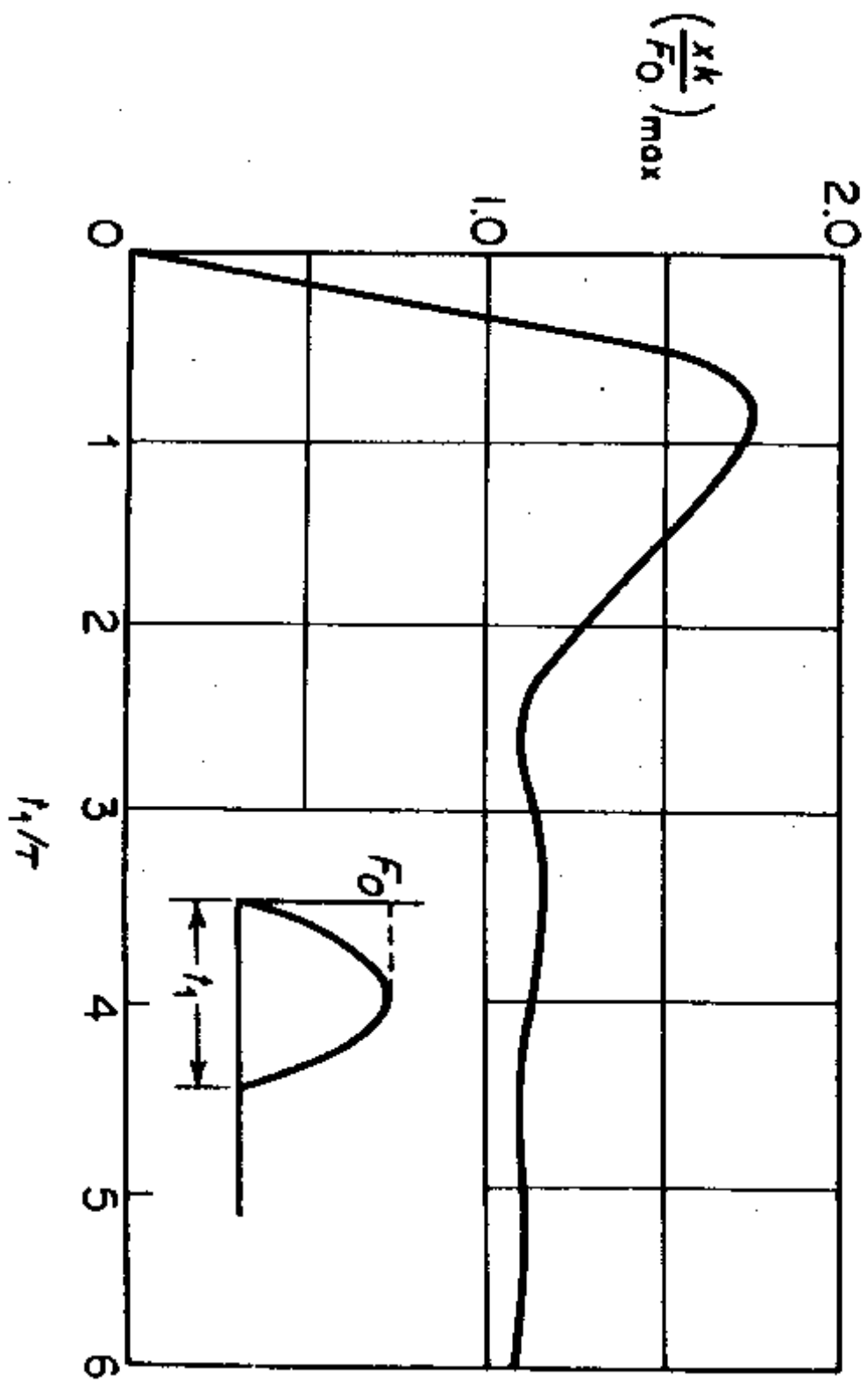
Fig. 4.5-4. Impulsive doublet.



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Hence, $\sin \omega_n t_p$ can be calculated from

$$\sin \omega_n t_p = -\sqrt{\frac{1}{2}(1 - \cos \omega_n t_1)} \tag{3.82}$$

and

$$\cos \omega_n t_p = \frac{-\sin \omega_n t_1}{\sqrt{2(1 - \cos \omega_n t_1)}} \tag{3.83}$$

Substitution of this expression into solution (3.74) evaluated at t_p yields, after some manipulation [here $x(t_p) = x_{max}$],

$$\frac{x_{max} k}{F_0} = 1 + \frac{1}{\omega_n t_1} \sqrt{2(1 - \cos \omega_n t_1)} \tag{3.84}$$

where the left side represents the dimensionless maximum displacement. It is customary to plot the response spectrum (dimensionless) versus the dimensionless frequency

$$\frac{t_1}{T} = \frac{\omega_n t_1}{2\pi} \tag{3.85}$$

where T is the structure's natural period. This provides a scale related to the characteristic time, t_1 , of the input. Figure 3.16 is a plot of the response spectrum for the ramp input force of Figure 3.15. Note that each point on the plot corresponds to a different rise time, t_1 , of the excitation. The vertical scale is an indication of the relationship between the structure and the rise time of the excitation.

The response is plotted using equation (3.78) along with the maximum magnitude as given by equation (3.81) and the ramp input function in Figure 3.17. Note from these plots that the amplitude of the response is magnified, or larger than the level of the input force. If t_1 is chosen to be near a period (the minimum in Figure 3.16), then the response is lower than this value and the maximum response will be equal to the input level. The effects of the various parameters form the topic of shock isolation (Section 5.2, which can be read now) and are examined numerically in Section 3.8.

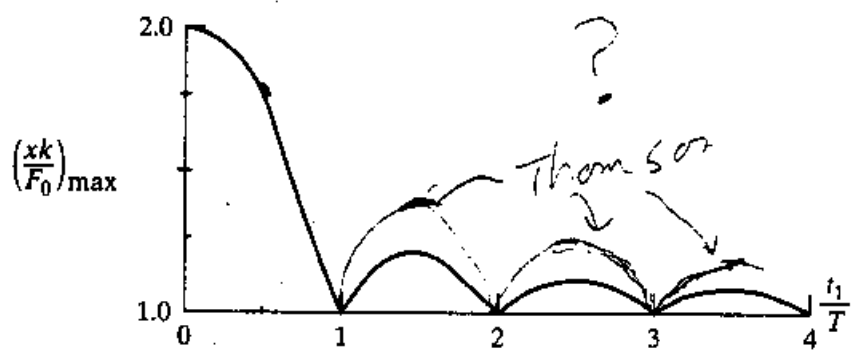


Figure 3.16 Response spectrum for the input force of Figure 3.15. The vertical axis is the dimensionless maximum response, and the horizontal axis is the dimensionless frequency (or delay time).

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